MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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SELF ALIGNMENT TECHNIQUES FOR STRAPDOWN INERTIAL NAVIGATION SYSTEMS WITH AIRCRAFT APPLICATION

by

Kenneth R. Britting and Thorgeir Palsson

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Director

ABSTRACT

One of the more critical problem areas in the application of strapdown inertial techniques to the navigation of commercial aircraft is that of initial alignment. A two stage self-alignment scheme which appears promising in this regard is explored. The first or "coarse" alignment stage utilizes the measurement of the gravity and earth rotation vectors to directly compute the transformation matrix relating the body frame to a reference frame. A linearized error analysis is presented. The second "fine" alignment stage corrects the initial estimate of the transformation by feeding back a computed angular velocity command to the transformation computer. This correction signal is computed by using estimates of the error angles between a known reference frame and the corresponding computed frame. Kalman filtering techniques are used to implement this technique and an error analysis is presented.

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SECTION I

INTRODUCTION

The current interest in analytic or strapdown inertial navigation systems for aircraft application is directly attributable to the revolution in computer technology which now allows solution of the navigation equations with a machine whose physical characteristics are compatible with aircraft requirements. The concepts involved in the design of such a system have been thoroughly explored in the literature (see Refs. 1 and 2 for instance), while the error analysis and detailed application is a subject of current research (Refs. 3, 4, and 5). Analytic systems offer the possibility of higher reliability than present gimballed systems since redundancy can be provided at the component level rather than at the subsystem level. Thus, the analytic inertial navigation system, augmented by appropriate navigation aids to bound the inherent long term error, is receiving serious consideration for application in the advanced supersonic transport.

One of the more critical problem areas in this application is that of initial alignment within the environment and time constraints imposed by commercial aircraft operation. That is, the system must be aligned within the necessary tolerances in the short period of time necessary for commercial success of the aircraft in the face of deleterious motions of the aircraft caused by wind gusts, the loading of passengers and cargo, fuel ingestion, etc. This paper will explore a two stage alignment technique which appears to be promising in this regard.

The problem of alignment in a strapdown inertial guidance system is basically that of determining the transformation matrix, which relates vectors in the instrumented body coordinate frame (b-frame) to the same vectors expressed in inertial coordinates (i-frame) or equivalently, in some other computation frame. Two methods for self-contained alignment will be considered here. Both of these use the fact that two vectors uniquely determine the transformation matrix between two coordinate frames, if they are known in both frames and are not colinear.

The two procedures are:

1. Analytic Alignment

   The transformation matrix is computed directly using the knowledge of $\underline{g}$ and $\underline{\omega}_{ie}$, i.e., the gravity and earth rotation vectors, in the two frames. These vectors are known in the inertial frame to the accuracy of the time measuring device used. The body frame coordinates are measured by the accelerometers and gyros and therefore contain instrument uncertainties.

2. Corrective Alignment

If it may be assumed that some estimate of the transformation matrix is available, then this initial matrix can be corrected by feeding an appropriate rate signal to the transformation computer. This correction signal can be computed by using estimates of the error angles between a known reference frame and the corresponding computed frame.

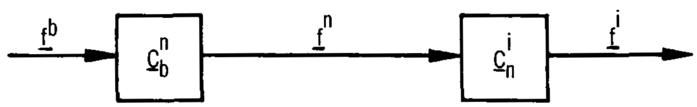
For either alignment scheme, the instrumented frame is taken to be stationary with respect to the earth, except for disturbances which occur due to wind gusts or loading and unloading of the vehicle in which the system is mounted. The application to a supersonic transport airliner is of particular interest, since external alignment techniques would appear to be impracticable for that vehicle. However, no data is available at this time on the motion of such or a similar aircraft due to wind gusts and other disturbances on the ground. The two methods will now be treated in detail.

SECTION II

ANALYTIC ALIGNMENT

A. Description

It is convenient to split the transformation between the b and i frames into two parts by using the local vertical or navigation coordinate system (n-frame), as an intermediate frame. The transformation of the specific force vector $\underline{f}$ can then be visualized as follows:



where the vector superscript refers to the coordinate system in which the vectors are resolved. The transformation $\underline{C}_n^i$, i.e. from n-frame to i-frame, is known as a function of time for any given latitude and longitude. The transformation $\underline{C}_b^n$ remains to be determined. The $\underline{C}_b^n$ matrix can be found by the estimation of two vectors; namely the earth rate vector, $\underline{\omega}_{ie}$, and the gravity vector, $\underline{g}$ in the two frames of interest. For the body frame these vectors are obtained by the gyros and the accelerometers respectively, and in the navigation frame they are known and constant, which makes this frame a convenient reference. The gravity and angular rate vectors transform according to the following expressions:

$$\underline{g}^b = \underline{C}_n^b \, \underline{g}^n$$

$$\underline{\omega}_{ie}^b = \underline{C}_n^b \, \underline{\omega}_{ie}^n$$

If $\underline{\nu}$ is defined as $\underline{\nu} = \underline{g} \times \underline{\omega}_{ie}$, we also have:

$$\underline{\nu}^b = \underline{C}_n^b \, \underline{\nu}^n$$

Since $\underline{C}_b^n = \left(\underline{C}_n^b \right)^{-1} = \left(\underline{C}_n^b \right)^T$ these three vector relations can be written:

$$\begin{bmatrix} (\underline{g}^n)^T \\ (\underline{\omega}_{ie}^n)^T \\ (\underline{\nu}^n)^T \end{bmatrix} \underline{C}_b^n = \begin{bmatrix} (\underline{g}^b)^T \\ (\underline{\omega}_{ie}^b)^T \\ (\underline{\nu}^b)^T \end{bmatrix}$$

or finally:

$$\underline{C}_b^n = \begin{bmatrix} (\underline{g}^n)^T \\ (\underline{\omega}_{ie}^n)^T \\ (\underline{\nu}^n)^T \end{bmatrix}^{-1} \begin{bmatrix} (\underline{g}^b)^T \\ (\underline{\omega}_{ie}^b)^T \\ (\underline{\nu}^b)^T \end{bmatrix}$$

Thus the alignment matrix is uniquely defined provided that the inverse indicated above exists. This inverse exists if no one row of the matrix is a linear combination of the remaining rows. This condition is always satisfied if the two vectors, $\underline{g}$ and $\underline{\omega}_{ie}$, are not collinear. These vectors coincide only at the earth's poles, where the analytic alignment procedure is useless. Kasper[(6)] shows that for fixed base alignment, the analytic scheme compares favorably with the existing optical alignment methods.

In the present case, however, its performance deteriorates because of the angular disturbance vibrations and accelerations. The effect is two-fold; first, the disturbances corrupt the measurements of $\underline{g}^b$ and $\underline{\omega}_{ie}^b$ since the measured quantities are:

$$\underline{f} = \underline{g} + \underline{f}_d$$

$$\underline{\omega}_{ib} = \underline{\omega}_{ie} + \underline{\omega}_d$$

where d indicates the disturbance quantities; secondly, $\underline{g}^b$ and $\underline{\omega}_{ie}^b$ become functions of time to some extent. This can be seen from the fact that since $\dot{\underline{\omega}}_{ie}^n = 0$,

$$\dot{\underline{\omega}}_{ie}^b = -\underline{\Omega}_{nb}^b \, \underline{\omega}_{ie}^b$$

where the elements of the skew symmetric matrix $\underline{\Omega}_{nb}^b$ are given by the components of $\underline{\omega}_d$. It is, therefore, necessary to introduce some filtering in order to reduce the effects of these vibrations. A simple low-pass filter could be used to obtain the average values of the measured quantities. This would tend to give the average alignment matrix. It is clear, however, that the instantaneous position of the body frame can vary considerably from its average position, depending upon the motion of the aircraft. As a result, a large initial misalignment could exist when the system is switched to the navigation mode of operation if only an average alignment were achieved. If the statistics of the aircraft vibrations were available, a more elaborate optimal filtering scheme sould be constructed. However, it could prove difficult to separate the perturbations of $\underline{\omega}_{ie}^{b}$ and $\underline{g}^{b}$ from the disturbances $\underline{\omega}_d$ and $\underline{f}_d$ by linear filtering, since it is very likely that these components contain the same frequencies. In addition some time lag would be introduced by the filter. The analytic alignment method is therefore mainly useful as an average alignment, which is a rapid way of obtaining an initial estimate of the transformation matrix.

B. Error Analysis

An error anaylsis for this alignment scheme, which takes into account the effect of instrument uncertainties and base motion is not readily amenable to analytic methods. The analysis which follows is intended to indicate an approach which will result in equations which can be solved on a digital computer. The equation for $\underline{C}_b^n$ can be written in the form:

$$\underline{C}_b^n = \underline{MQ}$$

where

$$\underline{M} = \begin{bmatrix} (\underline{g}^n)^T \\ (\underline{\omega}_{ie}^n)^T \\ (\underline{\nu}^n)^T \end{bmatrix}^{-1} \quad \text{and} \quad \underline{Q} = \begin{bmatrix} (\underline{g}^b)^T \\ (\underline{\omega}_{ie}^b)^T \\ (\underline{\nu}^b)^T \end{bmatrix}$$

The elements of $\underline{M}$ are constants in this case, but $\underline{Q}$ contains the measurement and instrument uncertainties. The above equation can be rewritten:

$$\underline{C}_{b'}^n = \underline{M}\ (\underline{Q} + \delta\underline{Q})$$

where b' indicates the computed body frame and δQ is the 3 x 3 uncertainty matrix. Thus:

$$\underline{C}_{b'}^n = \underline{C}_b^n\ \underline{C}_{b'}^b = \underline{C}_b^n[\underline{I} + \underline{Q}^{-1}\ \delta\underline{Q}]$$

which shows that

$$\underline{C}_{b'}^b = [\underline{I} + \underline{Q}^{-1}\ \delta\underline{Q}]$$

If the lengths of the measure vectors $\underline{g}^b$, $\underline{\omega}^b_{ie}$, and $\underline{\nu}^b$ and the angles between them are required to be constant, then the matrix $\underline{Q}^{-1}\delta\underline{Q}$ must necessarily be of the skew symmetric form:

$$\underline{Q}^{-1}\delta\underline{Q} = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}$$

where b_x, b_y, and b_z are the misalignment angles between the b and the b' frames about the x, y, and z body axes, respectively.

The b's are, of course, directly related to the instrument uncertainties. This relationship is found by direct evaluation of $\underline{Q}^{-1}\delta\underline{Q}$ where

$$\delta\underline{Q} = \begin{bmatrix} \delta\underline{g}^T \\ \delta\underline{\omega}^T \\ \delta\underline{\nu}^T \end{bmatrix}$$

A particularly simple result emerges if the body frame is taken to be aligned with the local geographic frame. In this case it is found that:

$$\underline{Q}^{-1}\delta\underline{Q} = \begin{bmatrix} \frac{\delta g_x}{g}\tan L + \frac{\delta\omega_x}{\omega_{ie}}\sec L & \frac{\delta g_y}{g}\tan L + \frac{\delta\omega_y}{\omega_{ie}}\sec L & \frac{\delta g_z}{g}\tan L + \frac{\delta\omega_z}{\omega_{ie}}\sec L \\ -\frac{\delta g_y}{g}\tan L - \frac{\delta\omega_y}{\omega_{ie}}\sec L & \frac{\delta\omega_x}{\omega_{ie}}\sec L + \frac{\delta g_x}{g} + \frac{\delta g_x}{g}\tan L & -\frac{\delta g_y}{g} \\ \frac{\delta g_x}{g} & \frac{\delta g_y}{g} & \frac{\delta g_z}{g} \end{bmatrix}$$

where L is the geographic latitude

It is seen that the matrix is not in the desired skew symmetric form. Moreover, it is difficult to directly apply the constraints that the lengths of the measured vectors and the angles between them be constant. This problem is resolved in a practical manner by requiring that the computed transformation, $\underline{C}^n_{b'}$, be orthogonal. This accomplished by setting:

$$\underline{C}^* = \underline{C}^n_{b'}[(\underline{C}^n_{b'})^T(\underline{C}^n_{b'})]^{-1/2}$$

where

$\underline{C}^* \sim$ optimal orthogonal approximation to $\underline{C}_{b'}^n$, in the sense that trace $(\underline{C}^* - \underline{C}_{b'}^n)^T(\underline{C}^* - \underline{C}_{b'}^n)$ is minimized.

Since

$$\underline{C}_{b'}^n = \underline{C}_b^n(\underline{I} + \underline{Q}^{-1}\delta\underline{Q}),$$

$$\underline{C}^* = \underline{C}_b^n[\underline{I} + \underline{Q}^{-1}\delta\underline{Q}][\underline{I} + \delta\underline{Q}^T(\underline{Q}^{-1})^T + \underline{Q}^{-1}\delta\underline{Q}]^{-1/2}$$

where products of error quantities have been neglected. If a series expansion is made of the square root term, there results:

$$\underline{C}^* = \underline{C}_b^n \{\underline{I} + \tfrac{1}{2}[\underline{Q}^{-1}\delta\underline{Q} - \delta\underline{Q}^T(\underline{Q}^{-1})^T]\}$$

The expansion in the square brackets is found to be skew symmetric, since

$$\underline{C}^* = \underline{C}_b^n \left[\underline{I} + \begin{bmatrix} 0 & -b_z^* & b_y^* \\ b_z^* & 0 & -b_x^* \\ -b_y^* & b_x^* & 0 \end{bmatrix} \right]$$

where

$$b_x^* = \frac{\delta g_y}{g}$$

$$b_y^* = -\frac{1}{2}\left(\frac{\delta g_x}{g} - \frac{\delta g_z}{g}\tan L - \frac{\delta\omega_z}{\omega_{ie}}\sec L\right)$$

$$b_z^* = -\tan L\,\frac{\delta g_y}{g} - \sec L\,\frac{\delta\omega_y}{\omega_{ie}}$$

Thus it is seen that one can expect to see a north level error of about 3.4 $\widehat{\text{min}}$ per milli -g east accelerometer uncertainty. The east level error due to north accelerometer uncertainty is 1.7 $\widehat{\text{min}}$/milli -g; to azimuth accelerometer uncertainty, 1.7 $\widehat{\text{min}}$ tanL/milli -g; and to azimuth gyro uncertainty 1.7 $\widehat{\text{min}}$ secL/meru. The azimuth error angle due to the east accelerometer uncertainty is given by -3.4 $\widehat{\text{min}}$/milli -g, and due to east gyro uncertainty is given by -3.4 $\widehat{\text{min}}$ secL/meru.

SECTION III

CORRECTIVE ALIGNMENT

A. Description

This procedure can be mechanized in the following manner, using the local vertical navigation frame as a reference frame. (See Fig. 1.) As before the

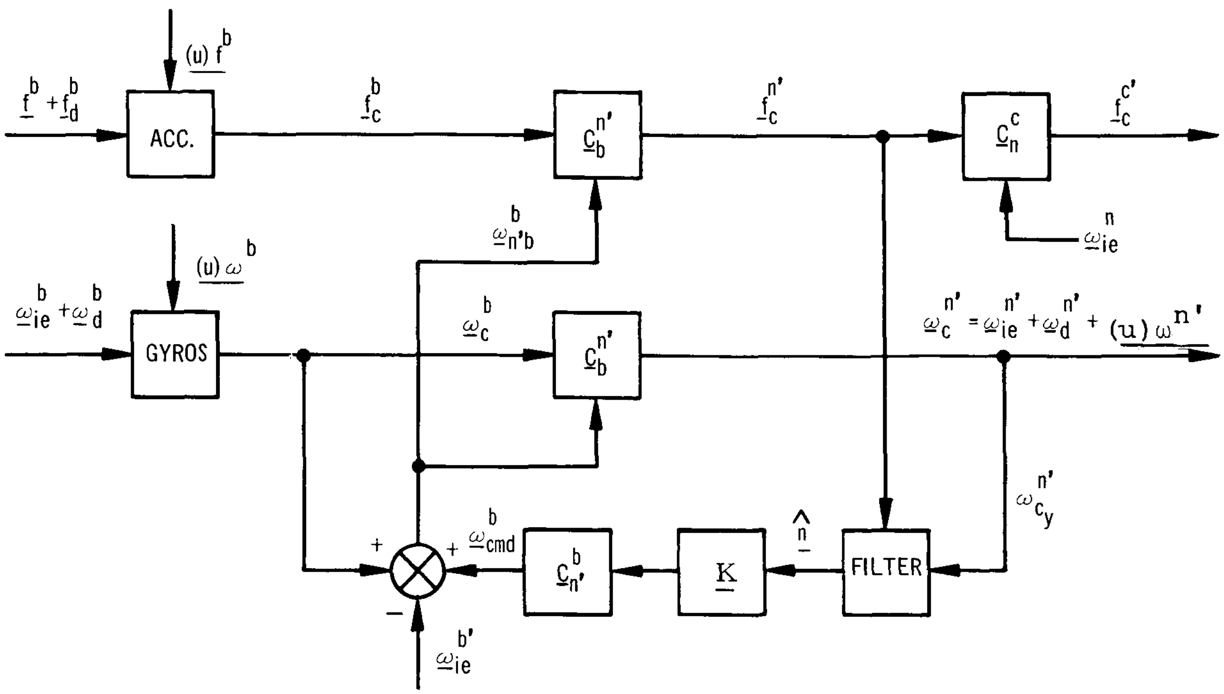


Fig. 1. Self corrective alignment scheme.

transformation consists of two steps; i.e. from the instrumented body frame, "b", to the navigation frame, "n", and then to the inertial frame used for computations.

It is assumed here that an initial estimate of the transformation matrix is available. This corresponds to a small angle misalignment of the computed and actual reference frame. The method, basically, consists of detecting the error angles between these two frames via the processed accelerometer and gyro signals and generating a signal to the transformation computer in order to reduce these angles as close to zero as possible. At the same time some compensation must be provided for the disturbance angular vibrations similar to the base isolation of a gimballed platform system. The transformation matrix $\underline{C}_b^{n'}$ is updated using the relation:

$$\dot{\underline{C}}_b^{n'} = \underline{C}_b^{n'} \underline{\Omega}_{n'b}^b \tag{1}$$

where $\underline{\Omega}_{n'b}^b$ is a skew symmetric matrix of the angular velocity $\underline{\omega}_{n'b}^b$, which is fed to the transformation computer. This angular velocity signal would ideally be:

$$\underline{\omega}_{n'b}^b = \underline{\omega}_{cmd}^b + \underline{\omega}_d^b \tag{2}$$

where $\underline{\omega}_{cmd}$ is the computed correction signal and $\underline{\omega}_d^b$ compensates for the vibrations of the instrumented body frame. In Fig. 1, $\underline{\omega}_d^b$ is obtained by subtracting $\underline{\omega}_{ie}^{b'}$ from the total angular velocity, but since $\underline{\omega}_{ie}^{b'}$ is not equal to $\underline{\omega}_{ie}^b$ and the signal, in addition, contains the gyro drift $(U)\,\underline{\omega}^b$, $\underline{\omega}_{n'b}^b$ becomes:

$$\underline{\omega}_{n'b}^b = \underline{\omega}_{cmd}^b + \underline{\omega}_d^b + \underline{B}\,\underline{\omega}_{ie}^b + (U)\,\underline{\omega}^b \tag{3}$$

Here, $\underline{B}$ is defined as before as the antisymmetric matrix of the misalignment angles between the actual and computed body frames, b and b'.

$$\underline{B} = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}$$

B. Error Analysis

The error angle equations can now be derived: Substituting the skew symmetric form of Eq. (3) into Eq. (1) yields:

$$\dot{\underline{C}}_b^{n'} = \underline{C}_b^{n'}\underline{\Omega}_{cmd}^b + \underline{C}_b^{n'}\underline{\Omega}_d^b + \underline{C}_b^{n'}\,\delta\underline{\Omega}_{ie}^b + \underline{C}_b^{n'}(u)\,\underline{\Omega}^b \tag{4}$$

where $\delta\underline{\Omega}_{ie}^b$ is the skew symmetric form of $\underline{B}\,\underline{\omega}_{ie}^b$. Noting that

$$\dot{\underline{C}}_b^{n'} = \underline{C}_n^{n'}\dot{\underline{C}}_b^n + \dot{\underline{C}}_n^{n'}\underline{C}_b^n$$

and

$$\dot{\underline{C}}_b^n = \underline{C}_b^n\underline{\Omega}_d^b, \tag{5}$$

since the "b" frame rotates with an angular velocity of $\underline{\omega}_d$ with respect to the "n" frame, Eq. (4) then becomes:

$$\dot{\underline{C}}_n^{n'} \underline{C}_b^n = \underline{C}_b^{n'} \underline{\Omega}_{cmd}^b + \underline{C}_b^{n'} \delta \underline{\Omega}_{ie}^b + \underline{C}_b^{n'} (u) \underline{\Omega}^b \tag{6}$$

Using the fact that $\underline{C}_n^{n'} = \underline{I} - \underline{N}$, where $\underline{N}$ is the skew symmetric matrix of misalignment angles between "n" and "n'", and $\underline{N} = \underline{C}_b^n \underline{B} \underline{C}_n^b$, Eq. (6) reduces to:

$$\dot{\underline{B}} = -\underline{\Omega}_{cmd}^b - (u) \underline{\Omega}^b - \delta \underline{\Omega}_{ie}^b \tag{7}$$

or equivalently for $\underline{N}$:

$$\dot{\underline{N}} = -\underline{\Omega}_{cmd}^n - (u) \underline{\Omega}^n - \delta \underline{\Omega}_{ie}^n \tag{8}$$

where higher order terms have been neglected.

In order to drive $\underline{N}$ to zero, $\underline{\omega}_{cmd}^{n'}$ can be chosen to be a linear function of the measured estimate of $\underline{N}$. The vector form of Eq. (8) then becomes:

$$\dot{\underline{n}} + \underline{K}\hat{\underline{n}} = -(u) \underline{\omega}^n - \underline{\Omega}_{ie}^n \underline{n} \tag{9}$$

where $\underline{K}$ remains to be specified, $\hat{\underline{n}}$ indicates an estimate of $\underline{n}$; $[\hat{\underline{n}} = \underline{n} + \delta \underline{n}]$. It is noted that the three scalar equations are coupled through the term $\underline{\Omega}_{ie}^n \underline{n}$.

The error angle vector, $\underline{n}$, must now be measured. A direct indication of the three components can be obtained from the computed horizontal components of $\underline{g}$ and the computed east component of $\underline{\omega}_{ie}$, which are approximately proportional to n_x, n_y, and n_z. Specifically, since

$$\underline{f}_c^{n'} = \underline{C}_b^{n'} \underline{f}_c^b = (\underline{I} - \underline{N}) \underline{f}_c^n = (\underline{I} - \underline{N}) [\underline{f}^n + \underline{f}_d^n + (u) \underline{f}^n]$$

and

$$\underline{f}^n = \{ 0, \quad 0, \quad g \}$$

then

$$f_{x_{n'}} = -n_y g + \delta f_{x_n} \tag{10a}$$

$$f_{y_{n'}} = n_x g + \delta f_{y_n} \tag{10b}$$

where δf_{x_n} and δf_{y_n} represent the uncertainties in the computed north and east specific force components which are caused by the accelerometer uncertainties and the disturbance accelerations. In a similar fashion the computed east component of earth rates is given by:

$$\omega_{y_{n'}} = -\omega_{ie} \cos L\,(n_z + n_x \tan L) + \delta\omega_{y_n} \tag{10c}$$

where L is the geographic latitude and $\delta\omega_{y_n}$ represents the uncertainty in the computed east component of earth rate caused by the gyro uncertainties and the disturbance angular velocity. This arrangement for the extraction of $\underline{n}$ is shown in Fig. 1.

Alternatively, the vector products of the measured and actual vectors, $\underline{g}^n \times \underline{g}_m^{n'}$ and $\underline{\omega}_{ie}^n \times \underline{\omega}_{ie_m}^{n'}$, could be used to indicate the misalignment. It is now necessary to determine the form of the $\underline{K}$ matrix in Eq. (9). Assuming for a moment that no disturbance errors exist, Eq. (9) becomes:

$$\dot{\underline{n}} = \underline{\Omega}_{ie}^n \underline{n} - \underline{K}\underline{n} \tag{9a}$$

which can be identified with the general form:

$$\dot{\underline{x}} = \underline{F}\,\underline{x} + \underline{u} \tag{10}$$

where

$$\underline{x} = \underline{n}$$

$$\underline{F} = \underline{\Omega}_{ie}^n$$

$$\underline{u} = -\underline{K}\,\underline{n}$$

One way of determining $\underline{K}$ is to define a cost function of the form

$$J = \frac{1}{2} \int_{t_o}^{t_f} [\underline{x}^T \underline{U}\,\underline{x} + \underline{u}^T \underline{W}\underline{u}]\, dt \tag{11}$$

where t_o and t_f are the initial and final times of the process, respectively, and $\underline{U}$ and $\underline{W}$ are positive definite weighting matrices which will be chosen to be constant here. The task of $\underline{U}$ and $\underline{W}$ is to insure that $\underline{x}$ and $\underline{u}$ remain within acceptable levels. Bryson[7] shows that a reasonable choice for $\underline{U}$ and $\underline{W}$ is:

$$\underline{U} = \begin{bmatrix} \frac{1}{x_{1m}^2} & 0 & 0 \\ 0 & \frac{1}{x_{2m}^2} & 0 \\ 0 & 0 & \frac{1}{x_{3m}^2} \end{bmatrix} \quad \text{and} \quad \underline{W} = \begin{bmatrix} \frac{1}{u_{1m}^2} & 0 & 0 \\ 0 & \frac{1}{u_{2m}^2} & 0 \\ 0 & 0 & \frac{1}{u_{3m}^2} \end{bmatrix}$$

x_{i_m} and u_{i_m} are the upper bounds on the state and control and will be taken to be constant.

It can be shown [7] that in order to minimize the performance index, $\underline{K}$ should have the value:

$$\underline{K} = \underline{W}^{-1}\underline{S} \tag{12}$$

where $\underline{S}$ is the solution of the steady state Riccatti equation:

$$\underline{S}\,\underline{F} - \underline{F}^T\underline{S} + \underline{S}\,\underline{W}^{-1}\underline{S} - \underline{U} = \underline{0} \tag{13}$$

where

$$\underline{F} = \begin{bmatrix} 0 & -\omega_{ie_z} & \omega_{ie_y} \\ \omega_{ie_z} & 0 & -\omega_{ie_x} \\ -\omega_{ie_y} & \omega_{ie_x} & 0 \end{bmatrix}$$

Equation (13) is most easily solved by numerical methods.

Another way of determining $\underline{K}$ is to require that Eq. (9) become uncoupled. This can be done approximately since $\underline{\omega}_{ie}^n$ is constant for a given latitude, so by choosing the off-diagonal terms of $\underline{K}$ equal to the corresponding terms of the anti-symmetric matrix $\underline{\Omega}_{ie}^n$, Eq. (9) becomes:

$$\dot{\underline{n}} + \underline{K}_d\underline{n} = -(u)\,\underline{\omega}^n - \underline{K}\,\underline{\delta n} \tag{14}$$

where $\underline{K}_d$ is a diagonal matrix.

Because of the uncertainties in the measurements of $\underline{n}$, which are due to instrument uncertainties and vibrations of the instrument frame, there remains a weak coupling of the equations (14) due to the term $\underline{K}\,\underline{\delta n}$, but this effect is of secondary importance and will be neglected.

The effect of measurement uncertainties and gyro drift on the solution for $\underline{n}$ is then, neglecting the weak coupling effects:

$$\dot{\underline{n}} + \underline{K}_d\underline{n} = -(u)\,\underline{\omega}^n - \underline{K}_d\,\underline{\delta n} \tag{14a}$$

where $\underline{\delta n}$ represents the error in the estimate of $\underline{n}$.

If the statistics of the disturbances $\underline{\omega}_d$ and $\underline{f}_d$ were known, it would be possible to construct a filter which separates the signal from the noise in some optimum way. A low-pass filter which approximately averages the input, may attenuate $\underline{\omega}_d$

sufficiently, but since $(U)\omega_y$, $(U)f_x$ and $(U)f_y$ can be expected to be very nearly constant during the alignment there will be a steady state alignment error due to gyro drift and accelerometer bias.

C. Filtering

The main problem in the alignment process is to obtain sufficiently good estimates of the error angles. After such an estimate has been made, the transformation matrix can be corrected almost instantaneously (assuming small angles). This suggests that the updating could be performed discretely rather than continuously. The in-between intervals would then be used for the estimation process. The correction signals would consist of the angular rotations about each axis needed to align the two frames in question. The equation of the error angles in this case becomes approximately:

$$\underline{n}(t + \Delta t) = \underline{n}(t) + \underline{\Delta n}(t) \tag{15}$$

where

$$\Delta \underline{n}(t) = -\underline{K}_d \underline{n} \Delta t - (u)\, \underline{\omega}^n \Delta t + \Omega_{ie}^n \underline{n}(t) \Delta t$$

and $-\underline{K}_d \underline{n} \Delta t$ is the desired rotation. The length of the interval depends on how much time is needed to get a satisfactory estimate of the error angles. If, for example, $\omega_{d_y} = A \sin \omega t$, the average is zero, but the computed average would be:

$$\hat{\omega}_{d_y} = \frac{1}{T} \int_0^T A \sin \omega t \, dt = \frac{A}{2\pi} \frac{\tau}{T} [1 - \cos \omega T]$$

In this case T would therefore have to be long enough to bring this average reasonably close to zero. The filtering time thus more or less determines the alignment time. Note that even without any disturbances some filtering is necessary because of instrument noise.

Instead of a simple averaging procedure, a more complex optimum filtering technique could be used. If it may be assumed that the statistics of the disturbance vibrations can be produced by passing an uncorrelated Gaussian signal through an appropriate shaping filter, then an optimum linear filter (Kalman filter) and alignment controller could be constructed as shown in Fig. 2. Here $\underline{v}$ and $\underline{u}$ are white noise signals and the system is augmented to include the state of the necessary shaping filters in addition to the error angles $\underline{n}$. The system equation is simply:

$$\dot{\underline{x}} = \underline{F}\, \underline{x} + \underline{v}$$

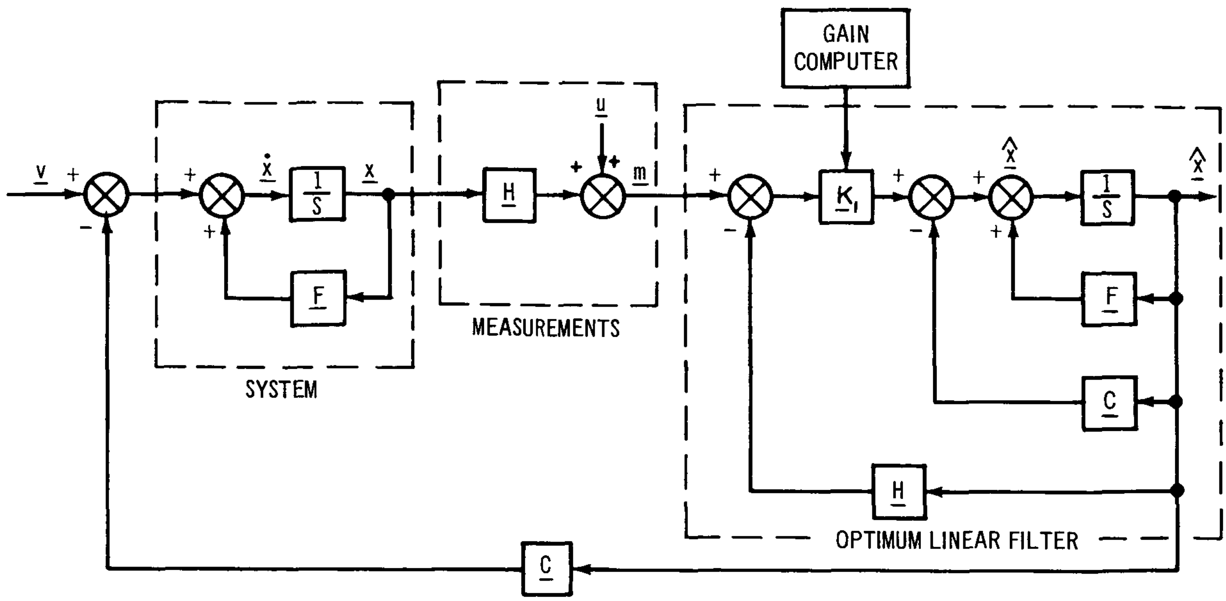


Fig. 2. Optimum linear filter.

and the measurements are given by:

$$\underline{m} = \underline{H}\,\underline{x} + \underline{u}$$

The gain matrix k must be computed continuously by numerical solution of the co-variance differential equation:

$$\dot{\underline{E}} = \underline{F}\,\underline{E} + \underline{E}\,\underline{F}^T + \underline{Q} \qquad (\underline{Q} = \overline{\underline{n}\,\underline{n}^T})$$

where

$$\underline{E} = \overline{(\hat{\underline{x}} - \underline{x})(\hat{\underline{x}} - \underline{x})^T} \qquad \text{and } E(0) \text{ is known}$$

Then the expression for $\underline{K}_1$ is:

$$\underline{K}_1 = \underline{E}\,\underline{H}^T(\underline{H}\,\underline{E}\,\underline{H}^T + \underline{R})^{-1}, \quad \text{where}(\underline{R} = \overline{\underline{u}\,\underline{u}^T})$$

The feedback matrix $\underline{C}$ is chosen in the same way as for the deterministic system. This procedure is valid, since it can be shown that the design of the optimum estimator and optimum controller are separable and independent for a linear system, which is excited by white noise.[8] It is clear that this technique requires a considerable amount of computations in order to adjust the filter gain, $\underline{K}_1$, continuously. There seems to be little point in pursuing this approach any further until some information is available on the vibration statistics and a realistic comparison of the different filtering schemes can be made.

D. Alignment Time and Accuracy

Without any uncertainty in the measurements of the misalignment angles and with decoupled error channels the governing equations are:

$$\dot{n}_j + k_j n_j = -(u)\,\omega_j^n \qquad j = x, y, z$$

The time constant is $\tau_j = 1/k_j$. k_j would be chosen as large as practical in this case and the alignment time could be made very short. A large k_j also gives a small steady state value of n_j since:

$$n_{j_{ss}} = -\frac{(u)\,\omega_j^n}{k_j}$$

However n_j cannot be obtained without uncertainty in a practical situation. The alignment time therefore mainly depends on the time needed to make a satisfactory estimate of n_j and to a lesser extent on the initial value of n_j. The time needed for the estimation on the other hand, depends upon the frequency content of the measurement noise. If only the constant component of the uncertainty is taken into account, the uncertainty in the azimuth error measurement is given by:

$$\delta n_z = \frac{(u)\,\omega_y}{\omega_{ie}\cos L} + \tan L\left[\frac{\delta f_y}{g} - \frac{(u)\,\omega_x}{k_x}\right]$$

and the total steady-state azimuth error becomes:

$$n_{z_{ss}} = -\frac{(u)\,\omega_z}{k_z} - \frac{(u)\,\omega_y}{\omega_{ie}\cos L} - \tan L\left[\frac{\delta f_y}{g} - \frac{(u)\,\omega_x}{k_y}\right]$$

The level errors can be found in a similar way and the steady state errors due to accelerometer bias and gyro drift are:

$$n_{x_{ss}} = -\frac{(u)\,\omega_x}{k_x} + \frac{(u)\,f_y}{g}$$

$$n_{y_{ss}} = -\frac{(u)\,\omega_y}{k_x} - \frac{(u)\,f_x}{g}$$

SECTION IV

CONCLUSIONS

It has been shown that the analytic method of alignment is mainly useful for obtaining an average transformation between the two frames. Since no linearization has been used, this method is well suited for calculating an initial estimate of the transformation matrix.

The second method, which has been analyzed by linearizing about the true alignment position is better suited for fine alignment, since compensation for the disturbances is provided in addition to the alignment control.

In both cases the alignment time is dependent on the time it takes to filter the measurement noise satisfactorily from the signal. Analogous to the physical gyrocompass, the accuracy that can be achieved is determined by the gyro drift and accelerometer bias.

A logical way of combining these two methods is to use the first one to compute an approximate initial alignment matrix which is then refined by the second method.

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